A multipoint stress mixed finite element method for linear elasticity

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1. Linear Elasticity Model

- Surface water - groundwater flow
- Flow in fractured porous media
- Flow through industrial filters, fuel cells
- Blood flow

The displacement field $\mathbf{u}$ and stress $\boldsymbol{\sigma}$ caused by a body force $\mathbf{f}$ acting on a linearly elastic body which occupies a region $\Omega \subset \mathbb{R}^d$ satisfy:

$$A\boldsymbol{\sigma} = \epsilon(\mathbf{u}), \quad \text{div}\boldsymbol{\sigma} = \mathbf{f}.$$

Compliance tensor: $A\boldsymbol{\sigma} = \dfrac{1}{2\mu}\left(\boldsymbol{\sigma} - \dfrac{\lambda}{2\mu + n\lambda}\text{tr}(\boldsymbol{\sigma})\mathbb{I}\right).$

Lamé coefficients: $\lambda(x),\ \mu(x).$

Rotation variable : $\mathbf{r} = \text{asym}(\nabla\mathbf{u})/2.$

$$\text{asym}(\boldsymbol{\tau}) = \begin{cases} \boldsymbol{\tau}_{12} - \boldsymbol{\tau}_{21},\ \boldsymbol{\tau} \in \mathbb{R}^{2\times2} \\ [\boldsymbol{\tau}_{32} - \boldsymbol{\tau}_{23},\ \boldsymbol{\tau}_{13} - \boldsymbol{\tau}_{31},\ \boldsymbol{\tau}_{21} - \boldsymbol{\tau}_{12}]^T,\ \boldsymbol{\tau} \in \mathbb{R}^{3\times3} \end{cases}$$

Formulation with weakly enforced symmetry :
Find $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{r}) \in H(\text{div}, \Omega; \mathbb{M}) \times L^2(\Omega, \mathbb{V}) \times L^2(\Omega, \mathbb{K})$

$$\begin{aligned} &(A\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\text{div}\ \boldsymbol{\tau}, \mathbf{u}) + (\text{asym}(\boldsymbol{\tau}), \mathbf{r}) = 0, && \boldsymbol{\tau} \in H(\text{div}, \Omega; \mathbb{M}) \\ &(\text{div}\ \boldsymbol{\sigma}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), && \mathbf{v} \in L^2(\Omega, \mathbb{V}) \\ &(\text{asym}(\boldsymbol{\sigma}), \mathbf{q}) = 0, && \mathbf{q} \in L^2(\Omega, \mathbb{K}). \end{aligned}$$

$\mathbb{M} = \mathbb{R}^{d\times d},\ \mathbb{V} = \mathbb{R}^d,\ \mathbb{K} = \mathbb{R}$ or $\mathbb{R}^d$

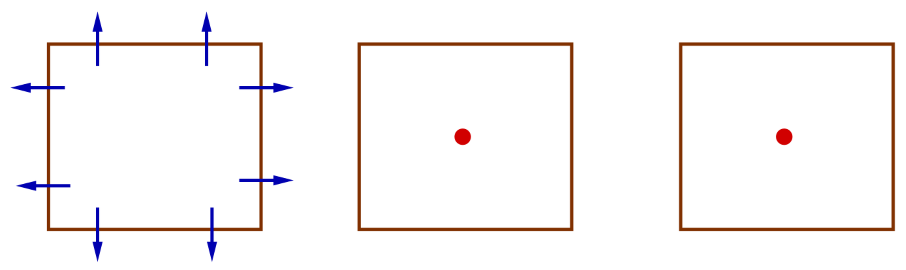
2. Multipoint stress mixed finite element method

- Based on MFE method with weak symmetry for simplicial elements in 2D and 3D and quadrilateral elements (h^2-parallelograms) in 2D.
- Spaces: $(\mathcal{BDM}_1)^d \times (\mathcal{P}_0)^d \times (\mathcal{P}_0)^{d/1}$ or $(\mathcal{BDM}_1)^d \times (\mathcal{P}_0)^d \times (\mathcal{P}_1)^{d/1}$
- Trapezoidal quadrature rule allows for local elimination of the stresses and rotations resulting in a cell-centered scheme for the displacements
- First order convergence for all variables in the natural norms
- Implemented on simplices in Fenics; quads in deal.II

Formulation with weakly enforced symmetry :
Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{r}_h) \in \Sigma_h \times \mathbf{V}_h \times \mathcal{Q}_h$ such that

$$\begin{aligned} &(A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\text{div}\ \boldsymbol{\tau}_h, \mathbf{u}_h) + (\text{asym}(\boldsymbol{\tau}_h), \mathbf{r}_h) = 0, && \boldsymbol{\tau}_h \in \Sigma_h \\ &(\text{div}\ \boldsymbol{\sigma}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), && \mathbf{v}_h \in \mathbf{V}_h \\ &(\text{asym}(\boldsymbol{\sigma}_h), \mathbf{q}_h) = 0, && \mathbf{q}_h \in \mathcal{Q}_h. \end{aligned}$$

$\Sigma_h \subset H(\text{div}, \Omega; \mathbb{M}),\ \mathbf{V}_h \subset L^2(\Omega, \mathbb{V}),\ \mathcal{Q}_h \subset L^2(\Omega, \mathbb{K})$



$\Sigma_h = (\mathcal{BDM}_1)^d, \quad \mathbf{V}_h = (\mathcal{P}_0)^d, \quad \mathcal{Q}_h = (\mathcal{P}_0)^{d/1}$

Figure 1. Mixed Finite Element Spaces

Multipoint stress mixed finite method 1

$L^2(\Omega)$ inner product: $(\cdot,\cdot)$

Trapezoidal quadrature rule: $(\cdot,\cdot)_Q$

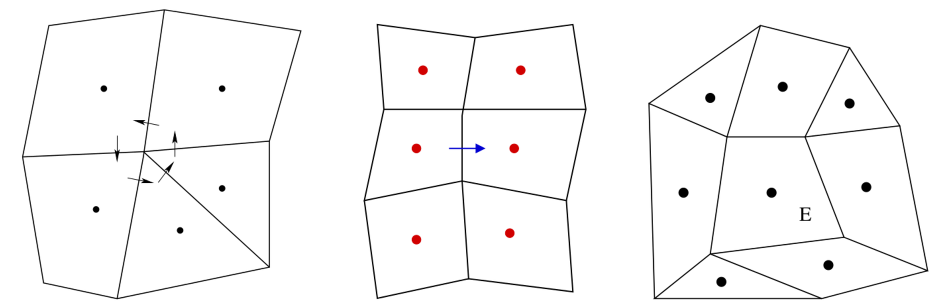
Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{r}_h) \in \Sigma_h \times \mathbf{V}_h \times \mathcal{Q}_h$ such that

$$\begin{aligned} &(A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_Q + (\text{div}\ \boldsymbol{\tau}_h, \mathbf{u}_h) + (\text{asym}(\boldsymbol{\tau}_h), \mathbf{r}_h) = 0, && \boldsymbol{\tau}_h \in \Sigma_h \\ &(\text{div}\ \boldsymbol{\sigma}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h,) && \mathbf{v}_h \in \mathbf{V}_h \\ &(\text{asym}(\boldsymbol{\sigma}_h), \mathbf{q}_h) = 0, && \mathbf{q}_h \in \mathcal{Q}_h. \end{aligned}$$

Reduction to a cell-centered method for $\mathbf{u}_h$ and $\mathbf{r}_h$.

$$\begin{pmatrix} A_{\boldsymbol{\sigma}\boldsymbol{\sigma}} & A^T_{\boldsymbol{\sigma}\mathbf{u}} & A^T_{\boldsymbol{\sigma}\mathbf{r}} \\ A_{\boldsymbol{\sigma}\mathbf{u}} & 0 & 0 \\ A_{\boldsymbol{\sigma}\mathbf{r}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}_h \\ \mathbf{u}_h \\ \mathbf{r}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ 0 \end{pmatrix}$$

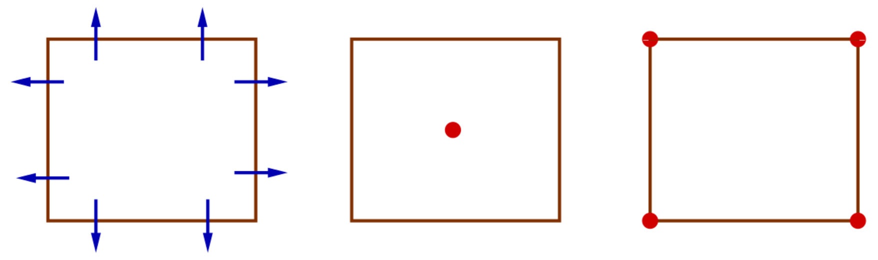
$A_{\boldsymbol{\sigma}\boldsymbol{\sigma}}$ is block-diagonal with blocks associated with vertices:



$$\boldsymbol{\sigma}_h = -A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}(A^T_{\boldsymbol{\sigma}\mathbf{u}}\mathbf{u}_h + A^T_{\boldsymbol{\sigma}\mathbf{r}}\mathbf{r}_h)$$

$$-\begin{pmatrix} A_{\boldsymbol{\sigma}\mathbf{u}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{u}} & A_{\boldsymbol{\sigma}\mathbf{u}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{r}} \\ A_{\boldsymbol{\sigma}\mathbf{r}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{u}} & A_{\boldsymbol{\sigma}\mathbf{r}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{r}} \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ \mathbf{r}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

Multipoint stress mixed finite method 2.



$\Sigma_h = (\mathcal{BDM}_1)^d, \quad \mathbf{V}_h = (\mathcal{P}_0)^d, \quad \mathcal{Q}_h = (\mathcal{Q}_1)^{d/1}$

Figure 2. Modified Mixed Finite Element Spaces

Find $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{r}) \in \Sigma_h \times V_h \times Q_h$ such that

$$\begin{aligned} &(A\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_Q + (\text{div}\ \boldsymbol{\tau}_h, \mathbf{u}_h) + (\text{asym}(\boldsymbol{\tau}_h), \mathbf{r}_h)_Q = 0, && \boldsymbol{\tau}_h \in \Sigma_h \\ &(\text{div}\ \boldsymbol{\sigma}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h,) && \mathbf{v}_h \in \mathbf{V}_h \\ &(\text{asym}(\boldsymbol{\sigma}_h), \mathbf{q}_h)_Q = 0, && \mathbf{q}_h \in \mathcal{Q}_h. \end{aligned}$$

The matrix $A_{\boldsymbol{\sigma}\mathbf{r}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{r}}$ becomes diagonal.

$$\mathbf{r} = -(A_{\boldsymbol{\sigma}\mathbf{r}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{r}})^{-1}A_{\boldsymbol{\sigma}\mathbf{r}}A^{-1}_{\boldsymbol{\sigma}\boldsymbol{\sigma}}A^T_{\boldsymbol{\sigma}\mathbf{u}}\mathbf{u}$$

3. Analysis of the MPSA FEM method

- inf-sup condition: there exists $\beta > 0$ such that

$$\inf_{(\mathbf{v}_h,\mathbf{q}_h)\in\mathbf{V}_h\times\mathcal{Q}_h}\ \sup_{\boldsymbol{\tau}_h\in\Sigma_h} \frac{(\text{div}\ \boldsymbol{\tau}_h, \mathbf{v}_h) + (\text{asym}(\boldsymbol{\tau}_h), \mathbf{q}_h)_Q}{\|\boldsymbol{\tau}_h\|_{\text{div}}(\|\mathbf{v}_h\| + \|\mathbf{q}_h\|)} \geq \beta$$

- continuity and coercivity of $(A\cdot,\cdot)$:

$$C_{\text{coer}}\|\boldsymbol{\tau}_h\|^2 \leq (A\boldsymbol{\tau}_h, \boldsymbol{\tau}_h), \quad (A\boldsymbol{\tau}_h, \kappa_{\mathbf{h}}) \leq \mathbf{C}_{\text{cont}}\|\boldsymbol{\tau}_{\mathbf{h}}\|\|\kappa_{\mathbf{h}}\|$$

- quadrature rule:
  $(A\cdot,\cdot)_Q$ is an inner product on Σ_h and $(A\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)^{1/2}_Q \sim \|\boldsymbol{\tau}_h\|$.
  $(\cdot,\cdot)_Q$ is an inner product on $\mathcal{Q}_h$ and $(\mathbf{q}_h, \mathbf{q}_h)^{1/2}_Q \sim \|\mathbf{q}_h\|$.

Theorem 1 *Solution of the MPSA FEM method satisfies:*

$$\|\boldsymbol{\sigma}_h\|_{H(div,\Omega)} + \|\mathbf{u}_h\|_{L^2(\Omega)} + \|\mathbf{r}_h\|_{L^2(\Omega)} \leq C_1\|\mathbf{f}\|_{L^2(\Omega)},$$

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{H(div,\Omega)} + \|\mathbf{u}_h - \mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{r}_h - \mathbf{r}\|_{L^2(\Omega)} \leq C_2 h\left(\|\boldsymbol{\sigma}\|_{H^1(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)} + \|\mathbf{r}\|_{H^1(\Omega)}\right)$$

where the constants C_1 and C_2 depend on λ, μ, β.

4. Numerical results

Test case 1: mild parameters. $\Omega = [0,1]^2,\ \lambda = 123,\ \mu = 79.3.$

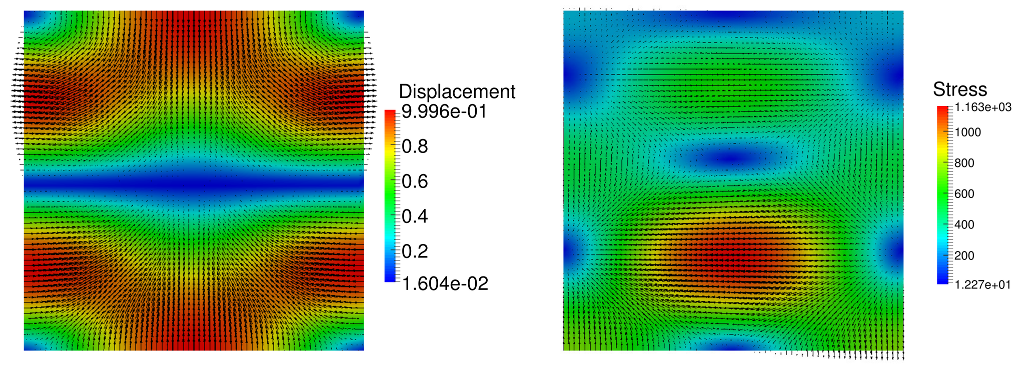


Figure 3. Test case 1. Displacement (left) and horizontal stress (right).

h	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ error	order	$\|\text{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)}$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order
1/8	1.36e-1		2.03e-1		1.76e-1		1.68e-1	
1/16	6.15e-2	1.2	1.02e-1	1.0	8.75e-2	1.0	5.37e-2	1.7
1/32	2.96e-2	1.1	5.19e-2	1.0	4.37e-2	1.0	1.66e-2	1.7
1/64	1.47e-2	1.0	2.67e-2	1.0	2.18e-2	1.0	5.26e-3	1.7
1/128	7.32e-3	1.0	1.40e-2	0.9	1.09e-2	1.0	1.73e-3	1.6

Table 1. Test case 1. Convergence on simplicies.

h	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ error	order
1/4	5.98E-01		5.35E-01		5.91E-01	
1/8	3.38E-01	0.82	3.11E-01	0.78	2.78E-01	1.09
1/16	1.38E-01	1.30	1.58E-01	0.98	1.37E-01	1.02
1/32	4.86E-02	1.50	7.89E-02	1.00	6.93E-02	0.98
1/64	1.66E-02	1.55	3.95E-02	1.00	3.50E-02	0.99

Table 2. Test case 1. Convergence on quads.

Test case 2: discontinuous force. $\Omega = [-1,1]^2,\ \lambda = \mu = 1$

$$\mathbf{f} = \begin{cases} (1.0,\ 0.0)^T & \text{for}(x - 0.5)^2 < 0.1^2 \text{ and } y^2 < 0.1^2 \\ (1.0,\ 0.0)^T & \text{for}(x + 0.5)^2 < 0.1^2 \text{ and } y^2 < 0.1^2 \\ (0.0,\ 1.0)^T & \text{for} x^2 < 0.1^2 \text{ and } y^2 < 0.1 \end{cases}$$

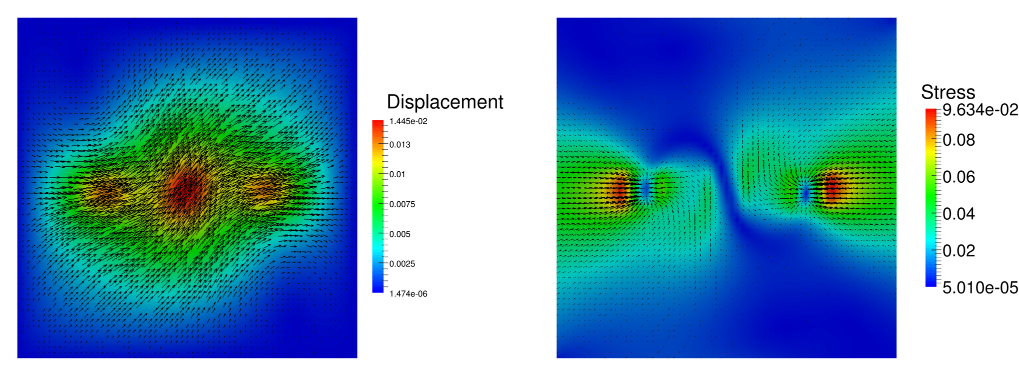


Figure 5. Test case 2. Displacement (left) and horizontal stress (right).

h	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ error	order	$\|\text{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)}$ error	order	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order
1/15	5.45E-01		1.17E+00		5.29E-01		4.80E-01	
1/30	2.25E-01	1.28	9.85E-01	0.24	1.35E-01	1.97	2.31E-01	1.06
1/60	1.24E-01	0.85	7.74E-01	0.35	6.39E-02	1.08	1.30E-01	0.82
1/120	4.54E-02	1.46	6.36E-01	0.28	2.27E-02	1.49	4.43E-02	1.56

Table 3. Test case 2. Convergence on simplicies.

Test case 3: heterogeneous media. $\Omega = [0,1]^2,\ \kappa = \dfrac{\lambda_1}{\lambda_2} = \dfrac{\mu_1}{\mu_2}$

$\kappa = 10^{-6}$ $\quad \kappa = 1$ $\quad \kappa = 10^6$

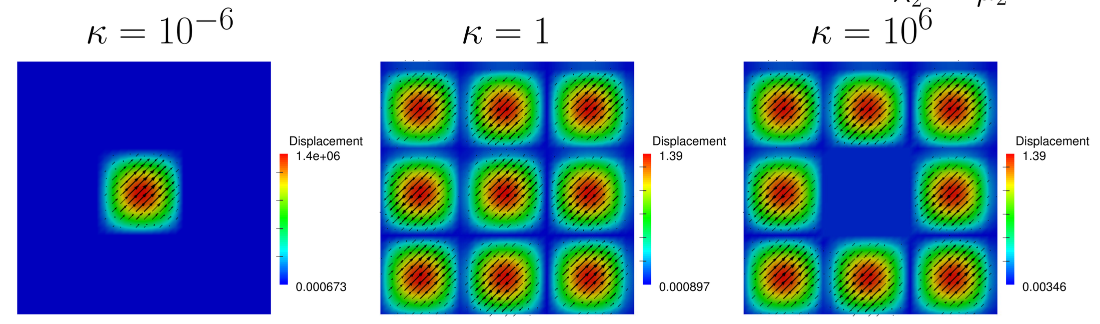


Figure 6. Test case 3. Displacement field.

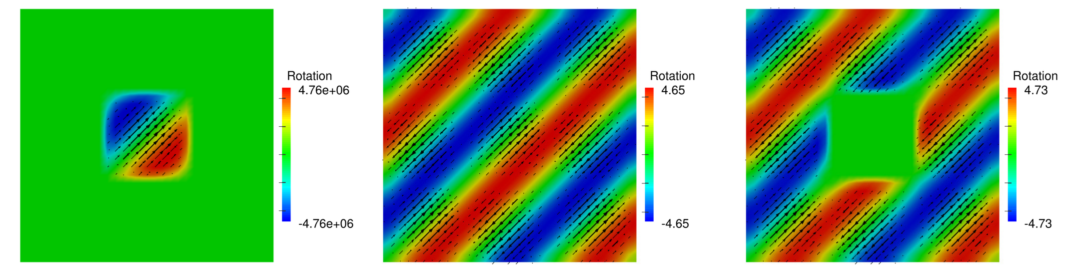


Figure 7. Test case 3. Displacement field over rotation.

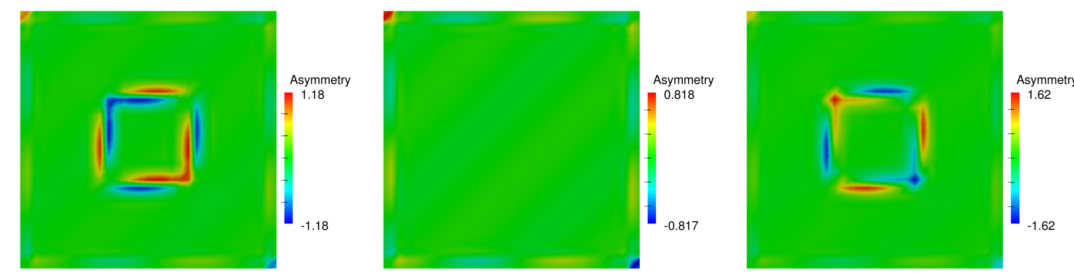


Figure 8. Test case 3. Asymmetry of stress.

h	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ error	order	$\|\text{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)}$ error	order
1/6	5.06E-01	–	6.09E-01	–	7.04E-01	–	7.28E-01	–
1/12	3.86E-01	0.4	2.88E-01	1.05	2.90E-01	1.29	3.33E-01	1.13
1/24	2.28E-01	0.76	1.64E-01	0.86	1.23E-01	1.23	1.58E-01	1.07
1/48	1.21E-01	0.91	1.04E-01	0.70	5.89E-02	1.07	7.79E-02	1.02
1/96	6.22E-02	0.96	6.76E-02	0.64	3.04E-02	0.95	3.88E-02	1.01

Table 4. Test case 3. Convergence on simplicies with $\kappa = 10^{-6}$.

h	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$ error	order	$\|\text{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2(\Omega)}$ error	order
1/6	7.39E-01	–	6.56E-01	–	7.39E-01	–	7.28E-01	–
1/12	3.44E-01	1.11	2.78E-01	0.96	3.20E-01	1.21	3.33E-01	1.13
1/24	1.67E-01	1.05	1.17E-01	1.17	1.43E-01	1.16	1.58E-01	1.07
1/48	8.32E-02	1	5.65E-02	1.02	7.30E-02	0.97	7.79E-02	1.02
1/96	4.17E-02	1	3.08E-02	0.86	3.96E-02	0.88	3.88E-02	1.01

Table 5. Test case 3. Convergence on simplicies with $\kappa = 10^6$.

Test case 4: 3D case. $\Omega = [0,1]^3,\ \lambda = \mu = 100$

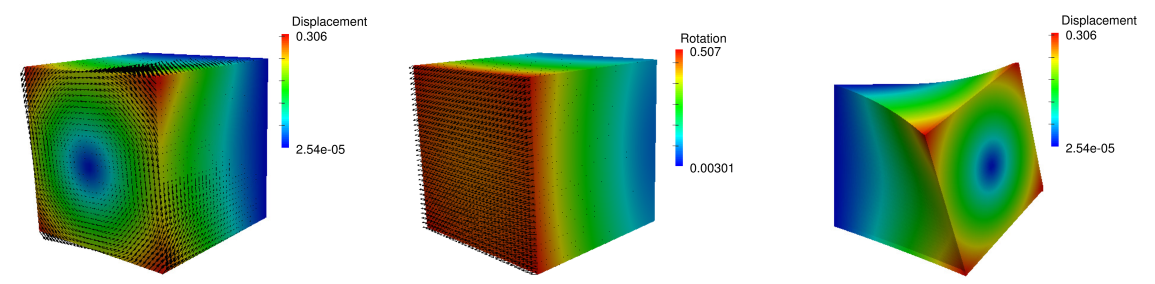


Figure 9. Test case 4. Displacement (left), rotation (center) and deformation (right)

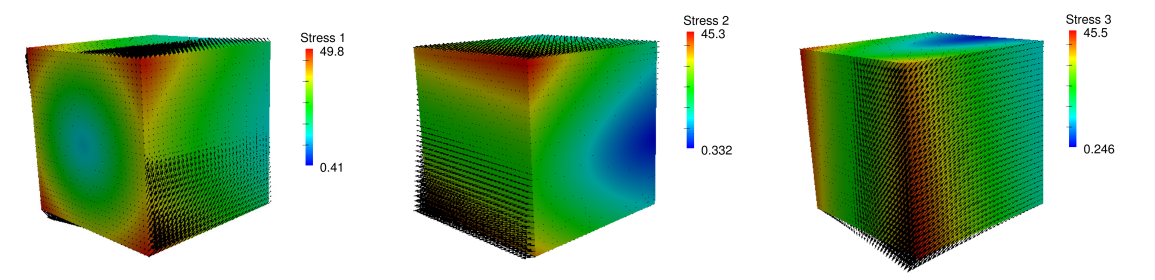


Figure 9. Test case 4. Stress: x- component (left), y- component (center) and z-component (right)

h	$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ error	order	$\|\mathbf{r} - \mathbf{r}_h\|_{L^2(\Omega)}$ error	order	$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{L^2(\Omega)}$ error	order	$\|\text{div}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_{L^2(\Omega)}$ error	order
1/2	4.197E-01		2.380E-01		5.400E-01		2.449E-01	
1/4	2.072E-01	1.02	1.005E-01	1.24	2.424E-01	1.16	1.208E-01	1.02
1/8	1.030E-01	1.01	3.929E-02	1.35	1.095E-01	1.15	6.020E-02	1.01
1/16	5.141E-02	1.00	1.472E-02	1.42	5.050E-02	1.12	3.009E-02	1.00
1/32	2.569E-02	1.00	5.382E-03	1.45	2.392E-02	1.08	1.718E-02	0.81

Table 6. Test case 4. Convergence on simplicies.

References

[1] I. Ambartsumyan, E. Khattatov, J. Nordbotten, and I. Yotov. A multipoint stress mixed finite element method for linear elasticity. In preparation.

[2] M. Wheeler and I. Yotov. A multipoint flux mixed finite element method. *SIAM J. NUMER. ANAL*, Vol. 44, No. 5, pp. 20822106.