A Lagrange multiplier method for flow in fractured poroelastic media

Eldar Khattatov†, Ivan Yotov†, Ilona Ambartsumyan†, Paolo Zunino‡

† Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania, USA;

‡ Department of Mechanical Engineering & Materials Science, University of Pittsburgh, Pittsburgh, Pennsylvania, USA;

1. Model Problem: Coupled Stokes-Biot System

- Groundwater flow and flow in fractured porous media
- Cardiovascular flow

Figure 1. Applications

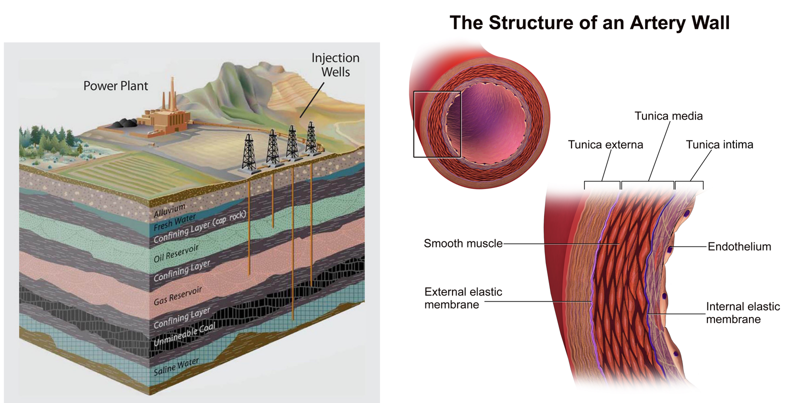
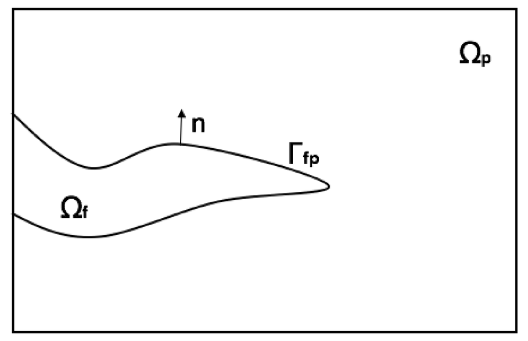


Figure 2. Stokes-Biot model



Biot system of poroelasticity in Ω_p:

$\mathbf{u}_p$ = Darcy velocity, p_p = pressure, $\boldsymbol{\eta}$ = displacement

$$-\nabla\cdot\boldsymbol{\sigma}_p(\boldsymbol{\eta}) = \mathbf{f}_p$$
$$\frac{\partial}{\partial t}(s_0 p_p + \alpha\nabla\cdot\boldsymbol{\eta}) + \nabla\cdot\mathbf{u}_p = q_p,$$
$$\boldsymbol{K}^{-1}\mathbf{u}_p = -\nabla p_p,$$

where $\boldsymbol{\sigma}_p(\boldsymbol{\eta}) = \lambda_p(\nabla\cdot\boldsymbol{\eta})\mathbf{I} + 2\mu_p\mathbf{D}(\boldsymbol{\eta}) - \alpha\mathbf{p_p}\mathbf{I}$ is a poroelastic stress tensor, α is Biot-Willis constant and $\boldsymbol{K}$ is a permeability tensor.

Stokes flow in Ω_f:

$\mathbf{u}_f$ = Stokes velocity, p_f = fluid pressure

$$\rho_f\frac{\partial\mathbf{u}_f}{\partial t} - \nabla\cdot\boldsymbol{\sigma}_f = \mathbf{f}_f,$$
$$\nabla\cdot\mathbf{u}_f = q_f,$$

where $\boldsymbol{\sigma}_f = -p_f\mathbf{I} + 2\mu_f\mathbf{D}(\mathbf{u}_f)$ is a Cauchy stress tensor.

Interface conditions on Γ_{fp}:

Mass conservation:

$\mathbf{u}_f\cdot\mathbf{n} = (\frac{\partial\boldsymbol{\eta}}{\partial t} + \mathbf{u}_p)\cdot\mathbf{n}$

Balance of normal fluid stress:

$-(\boldsymbol{\sigma}_f\mathbf{n})\cdot\mathbf{n} = p_p$

Continuity of momentum:

$\boldsymbol{\sigma}_f\mathbf{n} = \boldsymbol{\sigma}_p\mathbf{n}$

No slip or Beavers-Joseph-Saffman condition:

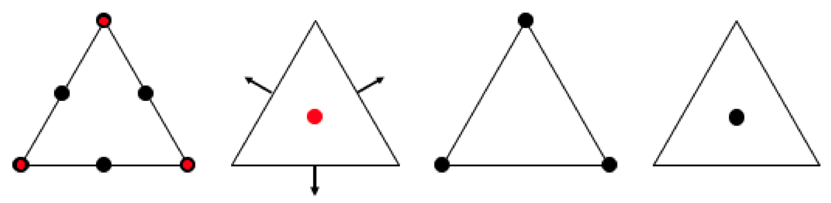
- $\mathbf{u}_f\cdot\boldsymbol{\tau} = \frac{\partial\boldsymbol{\eta}}{\partial t}\cdot\boldsymbol{\tau}$
- $-(\boldsymbol{\sigma}_f\mathbf{n})\cdot\boldsymbol{\tau} = c_{BJS}(\mathbf{u}_f - \frac{\partial\boldsymbol{\eta}}{\partial t})\cdot\boldsymbol{\tau}$

2. Discretization & numerical analysis

The discretization

- is derived on simplicial grids in 2D and 3D;
- uses standard conforming finite elements for Stokes and elasticity equations and mixed finite elements for Darcy;
- allows the fluid and poroelastic region grids to be non-matching through the interface;
- utilizes a Lagrange multiplier λ on the interface to weakly impose the continuity of flux;
- uses the space for λ being the normal trace of Darcy velocity space.

Figure 3. Example of lowest order FE spaces: $(\mathcal{P}_2\times\mathcal{P}_1)\times(\mathcal{RT}_0\times\mathcal{P}_0)\times\mathcal{P}_1\times\mathcal{P}_0$



Discrete weak fomulation

Find $(\mathbf{u}_{f,h}, p_{f,h}, \mathbf{u}_{p,h}, p_{p,h}, \boldsymbol{\eta}_{p,h}, \lambda_h)$ in $\mathbf{V}_{f,h}\times W_{f,h}\times\mathbf{V}_{p,h}\times W_{p,h}\times\mathbf{X}_{p,h}\times\Lambda_h$ such that

$$\begin{aligned}
&(\rho_f\partial_t\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_p^d(\mathbf{u}_{p,h}, \mathbf{v}_{p,h}) + a_{p,h}^e(\boldsymbol{\eta}_{p,h}, \boldsymbol{\xi}_{p,h})\\
&\quad + a_{BJS}(\mathbf{u}_{f,h}, \boldsymbol{\eta}_{p,h}; \mathbf{v}_{f,h}, \boldsymbol{\xi}_{p,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) + b_p(\mathbf{v}_{p,h}, p_{p,h})\\
&\quad + \alpha b_p(\boldsymbol{\xi}_{p,h}, p_{p,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \boldsymbol{\xi}_{p,h}; \lambda_h) = \mathbf{f}(\mathbf{v}_{\mathbf{f},\mathbf{h}}, \boldsymbol{\xi}_{\mathbf{p},\mathbf{h}}),\\
&(\partial_t s_0 p_{p,h}, w_{p,h}) - \alpha b_p(\partial_t\boldsymbol{\eta}_{p,h}, w_{p,h}) - b_p(\mathbf{u}_{p,h}, w_{p,h})\\
&\quad - b_f(\mathbf{u}_{f,h}, w_{f,h}) = q(w_{f,h}, w_{p,h}),\\
&b_\Gamma(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \partial_t\boldsymbol{\eta}_{p,h}; \mu_h) = 0,
\end{aligned}$$

With Backward Euler used for discrete time derivative

$$d_\tau u^n := \tau^{-1}(u^n - u^{n-1})$$

Convergence analysis of the discrete formulation

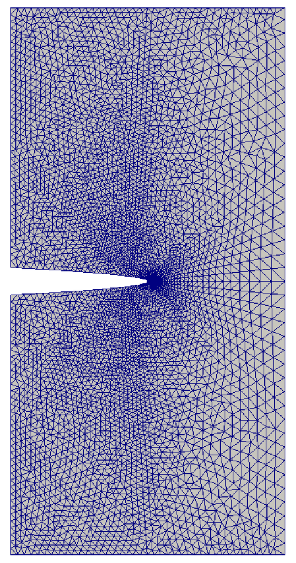
With k_f, s_f, k_p, s_p, k_s denoting the order of polynomials used in the discretization of Stokes, Darcy and elasticity equations respectively, there holds

$$\begin{aligned}
&\|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^\infty(L^2)} + \|\mathbf{u}_f - \mathbf{u}_{f,h}\|_{L^2(H^1)} + \|\mathbf{u}_p - \mathbf{u}_{p,h}\|_{L^2(L^2)}\\
&\quad + \|p_p - p_{p,h}\|_{L^\infty(L^2)} + \|\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h}\|_{L^\infty(H^1)}\\
&\quad + \sum_{j=1}^{d-1}\nu c_{BJS}\|\boldsymbol{K}_j^{-1/4}\left((\mathbf{u}_f - \mathbf{u}_{f,h}) - \partial_t(\boldsymbol{\eta}_p - \boldsymbol{\eta}_{p,h})\right)\cdot\boldsymbol{\tau}_{f,j}\|_{L^2(L^2(\Gamma_{fp}))}\\
&\le C\Big(h^{k_f}\|\mathbf{u}_f\|_{L^2(H^{k_f+1})} + h^{k_f}\|\mathbf{u}_f\|_{L^\infty(H^{k_f+1})} + h^{k_f}\|\partial_t\mathbf{u}_f\|_{L^2(H^{k_f+1})}\\
&\quad + h^{s_f+1}\|p_f\|_{L^2(H^{s_f+1})} + h^{k_p+1}\|\mathbf{u}_p\|_{L^2(H^{k_p+1})} + h^{k_s}\|\partial_t\boldsymbol{\eta}_p\|_{L^2(H^{k_s+1})}\\
&\quad + h^{k_p+1}\|\lambda\|_{L^2(H^{k_p+1}(\Gamma_{fp}))} + h^{k_s}\|\boldsymbol{\eta}_p\|_{L^\infty(H^{k_s+1})} + h^{k_p+1}\|\lambda\|_{L^\infty(H^{k_p+1}(\Gamma_{fp}))}\\
&\quad + h^{s_p+1}\|p_p\|_{L^\infty(H^{s_p+1})} + h^{k_p+1}\|\partial_t\lambda\|_{L^2(H^{k_p+1}(\Gamma_{fp}))} + h^{s_p+1}\|p_p\|_{L^2(H^{s_p+1})}\Big)
\end{aligned}$$

3. Convergence, BJS condition and continuity of flux

Convergence study on a reference domain

Figure 4. Reference grid



Parameter	Units	Value
Young's modulus	(Pa)	10^{10}
Fluid density	(kg/m^3)	1.0
Dynamic viscosity	(Pa s)	1.0
Lame coefficient	(Pa)	$5/12\cdot10^{10}$
Lame coefficient	(Pa)	$5/18\cdot10^{10}$
Hydraulic conductivity	(m^2/Pa s)	Id
Mass storativity	(Pa^{-1})	1.0
Biot-Willis constant		1.0
BJS coefficient		1.0
Total time	(s)	1.0

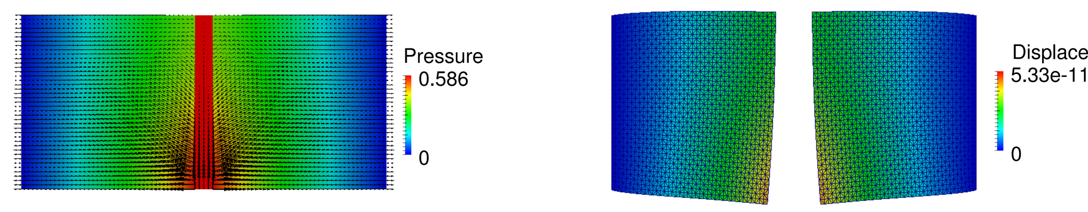
$\mathbf{u}_f\cdot\mathbf{n} = 10,\ \mathbf{u}_f\cdot\boldsymbol{\tau} = 0$ on Γ_{inflow}
$\mathbf{u}_p\cdot\mathbf{n} = 0,$ on Γ_{left}
$p_p = 0,$ on $\Gamma_{top}\cup\Gamma_{right}\cup\Gamma_{bottom}$
$\boldsymbol{\eta} = 0,$ on $\Gamma_{top}\cup\Gamma_{right}\cup\Gamma_{bottom}$

Table 1. Convergence study using the lowest order elements

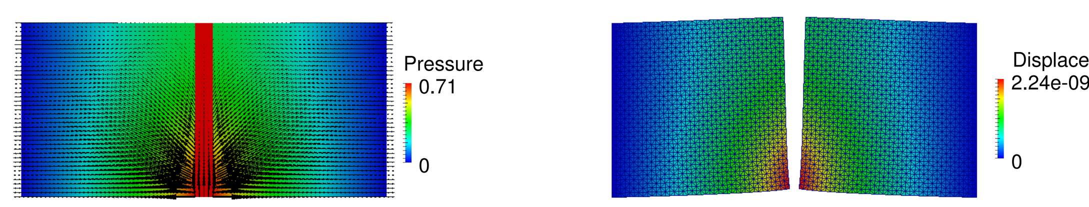
h	$\\|\mathbf{u}_f - \mathbf{u}_{f,h}\\|_{l^2(H^1)}$	rate	$\\|\mathbf{u}_f - \mathbf{u}_{f,h}\\|_{l^\infty(L^2)}$	rate
1/20	1.33e+00	0.00	9.58e-03	0.00
1/40	5.11e-01	1.38	1.87e-03	2.36
1/80	2.31e-01	1.15	3.88e-04	2.27
1/160	9.24e-02	1.32	7.07e-05	2.46
h	$\\|\mathbf{u}_p - \mathbf{u}_{p,h}\\|_{l^2(L^2)}$	rate	$\\|p_p - p_{p,h}\\|_{l^\infty(L^2)}$	rate
1/20	5.46E-02	—	4.37E-02	—
1/40	3.26E-02	0.74	2.04E-02	1.10
1/80	1.63E-02	1.00	8.74E-03	1.22
1/160	8.26E-03	0.98	2.91E-03	1.59
h	$\\|\boldsymbol{\eta} - \boldsymbol{\eta}_h\\|_{l^\infty(H_1)}$	rate		
1/20	1.07E-01	—		
1/40	5.95E-02	0.85		
1/80	2.92E-02	1.03		
1/160	1.05E-02	1.48		

Effect of BJS condition

Figure 5. Effect of BJS condition



Free slip $c_{BJS} = 0$



Slip with friction $c_{BJS} = 1$

Continuity of flux

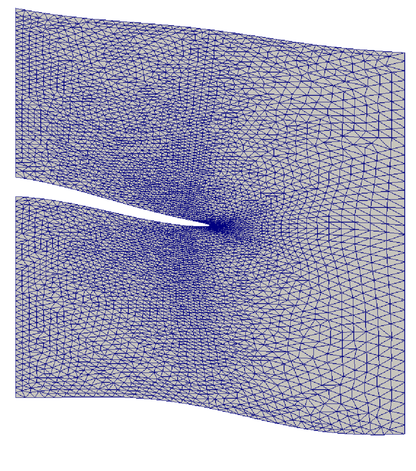
Table 2. Jump in fluxes across the interface, $\mathcal{R}_{\Gamma_{fp}} = \int_{\Gamma_{fp}}\left(\mathbf{u}_f\cdot\mathbf{n}_f + (\frac{\partial\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p)\cdot\mathbf{n}_p\right)$

	Lagrange multiplier		Nitsche	
h	$\mathcal{R}_{\Gamma_1}$	$\mathcal{R}_{\Gamma_2}$	$\mathcal{R}_{\Gamma_1}$	$\mathcal{R}_{\Gamma_2}$
1/20	4.44E-12	3.86E-12	2.75E-01	2.75E-01
1/40	1.97E-12	1.97E-12	4.87E-03	4.87E-03
1/80	4.23E-13	4.24E-13	1.54E-03	1.54E-03
1/160	1.07E-13	1.06E-13	3.85E-04	3.85E-04

4. Applications to reservoir simulation

Injection-production example

Figure 7. Physical grid



Parameter	Units	Value
Young's modulus	(KPa)	10^7
Fluid density	(kg/m^3)	897.0
Dynamic viscosity	(KPa s)	10^{-6}
Lame coefficient	(KPa)	$5/12\cdot10^7$
Lame coefficient	(KPa)	$5/18\cdot10^7$
Hydraulic conductivity	(m^2/KPa s)	$(200, 50)\cdot10^{-6}$
Mass storativity	(KPa^{-1})	6.89×10^{-2}
Biot-Willis constant		1.0
BJS coefficient		1.0
Total time	(s)	300

$\mathbf{u}_f\cdot\mathbf{n} = 10,\ \mathbf{u}_f\cdot\boldsymbol{\tau} = 0$ on Γ_{inflow}
$\mathbf{u}_p\cdot\mathbf{n} = 0,$ on Γ_{left}
$p_p = 1000,$ on $\Gamma_{top}\cup\Gamma_{right}\cup\Gamma_{bottom}$
$\boldsymbol{\eta} = 0,$ on $\Gamma_{top}\cup\Gamma_{right}\cup\Gamma_{bottom}$

Figure 8. Phases

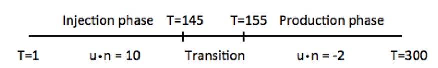
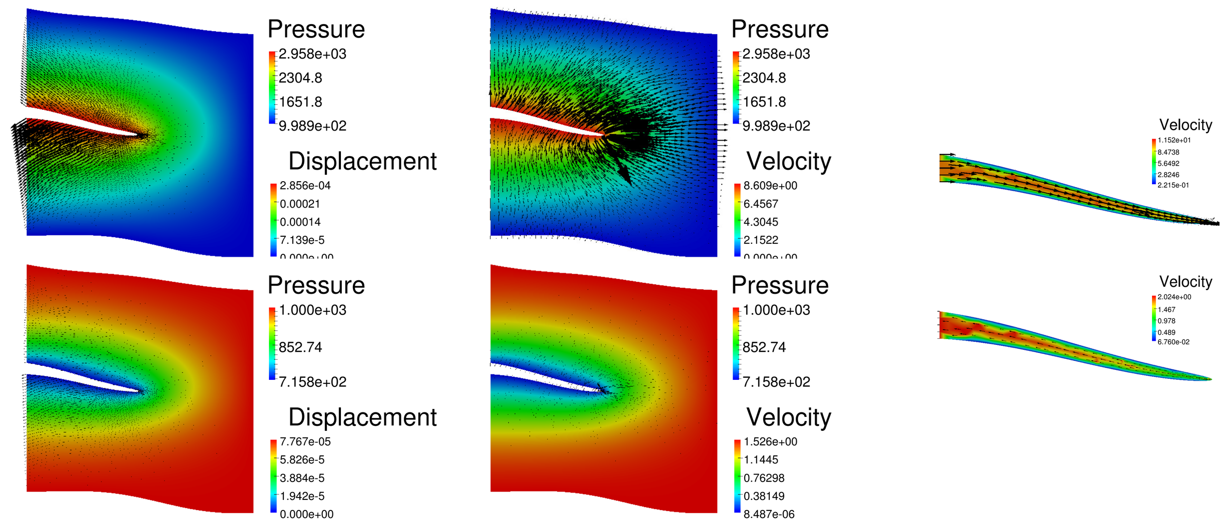


Figure 9. Last step of injection (top) and production (bottom) phases



Heterogeneous permeability example

Young's modulus: $E = E_0(1 - \frac{\phi}{0.5})^{2.1}$

Figure 10. Permeability, porosity and Young's modulus, SPE data.

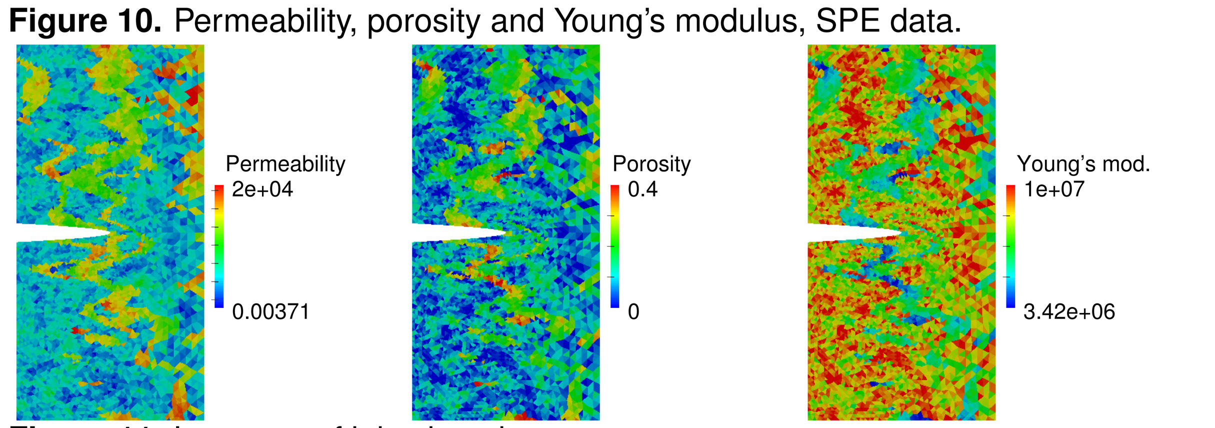
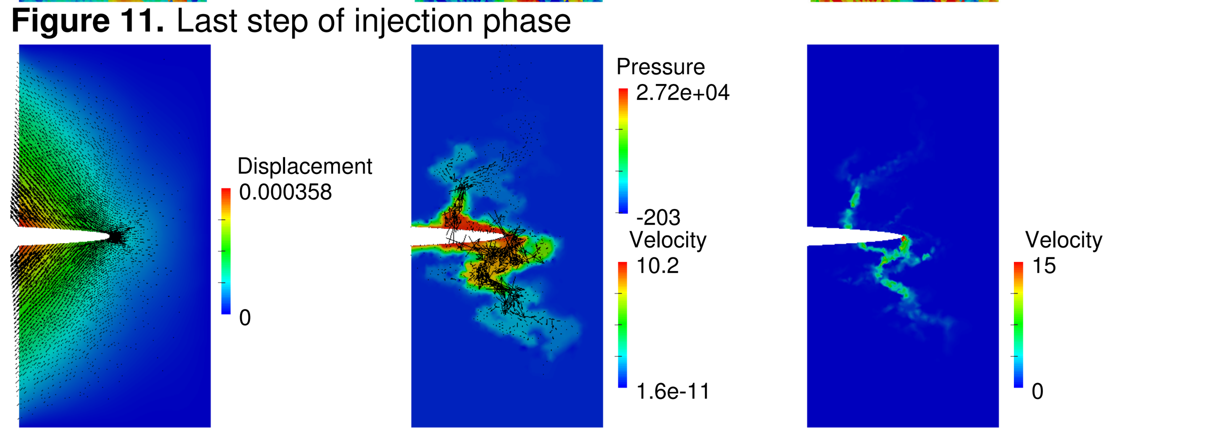


Figure 11. Last step of injection phase



5. References

[1] I. Ambartsumyan, E. Khattatov, I. Yotov and P. Zunino. A Lagrange multiplier method for flow in fractured poroelastic media. Preprint.

[2] M. Bukac, I. Yotov, R. Zakerzadeh and P. Zunino. Partitioning strategies for the interaction of a fluid with a poroelastic material based on a Nitsche's coupling approach. Comput. Methods Appl. Mech. Engrg. 292 (2015) 138170.